INVESTIGATION OF RENORMALIZATION GROUP METHODS FOR THE NUMERICAL SIMULATION OF ISOTROPIC TURBULENCE

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INTRODUCTION

Over the years, our research into turbulence at Edinburgh has concentrated on the application of renormalization methods to the prediction of the energy spectrum of isotropic turbulence. General discussions of this work will be found elsewhere (McComb 1990, 1995), while accounts of specific progress have been given previously in this conference series (McComb & Shanmugasundaram 1983, McComb, Filipiak, Roberts & Watt, 1991).

From a practical point of view, the most promising development in this area is undoubtedly **Renormalization Group** or **RG**. If we work in the Fourier representation, in principle, this involves the progressive averaging out of high-wavenumber modes in bands, with rescaling at each step, until a fixed point is reached. The result is, in effect, a 'subgrid model' for large-eddy simulation.

RG has enjoyed its successes in other areas of statistical physics. However, its application to turbulence faces several technical difficulties, which have to be circumvented by uncontrolled approximations. Indeed, in view of the deterministic nature of the Navier-Stokes equations, it is clear that the operation of averaging out the high-wavenumber modes while keeping the low-wavenumber modes constant, cannot be done rigorously and in itself can only be an approximation.

With points like this in mind, we have recently adopted direct numerical simulation as a tool for probing the basic feasibility of using RG techniques to reduce the number of degrees of freedom requiring to be numerically simulated. In this paper, we present some of the first results of this approach. We begin by discussing the RG approach in detail.

RENORMALIZATION GROUP THEORY

Basic Equations

Working in Fourier-wavevector ($\mathbf{k}$) space and restricting our attention to turbulent velocity fields which are homogeneous, isotropic and stationary, we may write the pair-correlation of velocities as

$$\langle u_\alpha(\mathbf{k},t)u_\alpha(\mathbf{k}',t')\rangle = Q(k,t-t')D_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}'), \quad (1)$$

where $Q(k,t-t')$ is the spectral density and the projector $D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta}+k_\alpha k_\beta k^{-2}$ arises due to the incompressibility condition. Thus, the energy spectrum $E(k) = 4\pi k^2 Q(k)$ with $Q(k) = Q(k,0)$ and the maximum cut-off wavenumber, k_0, is defined via the dissipation integral

$$\varepsilon = \int_0^\infty dk\ 2\nu_0 k^2 E(k) \simeq \int_0^{k_0} dk\ 2\nu_0 k^2 E(k), \quad (2)$$

where ε is the dissipation rate, ν_0 is the kinematic viscosity, and k_0 is of the same order of magnitude as the Kolmogorov dissipation wave-number.

Renormalization Group Theory

Taking our goal to be the calculation of the energy spectrum $E(k)$, our intermediate objective is to find an analytical method of reducing the number of degrees of freedom (or Fourier modes), in order to make the numerical solution of the equations of motion a practical proposition. Let us consider how this might be done by using RG.

First, we divide up the velocity field at $k = k_1$ as $u_\alpha(\mathbf{k},t) = u^-_\alpha(\mathbf{k},t)$ for $0 < k < k_1$ and $u_\alpha(\mathbf{k},t) = u^+_\alpha(\mathbf{k},t)$ for $k_1 < k < k_0$, where $k_1 = (1-\eta)k_0$ and the bandwidth parameter η satisfies the condition $0 < \eta < 1$. Working with the standard form of the solenoidal Navier-Stokes equation in k-space, we may write the evolution of the low-k velocity field for $0 < k < k_1$ as

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u^-_\alpha(\mathbf{k},t) = M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left[u^-_\beta(\mathbf{j},t)u^-_\gamma(\mathbf{k}-\mathbf{j},t) + 2u^-_\beta(\mathbf{j},t)u^+_\gamma(\mathbf{k}-\mathbf{j},t) + u^+_\beta(\mathbf{j},t)u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right], \quad (3)$$

and the evolution of the high-k velocity field for the first shell, $k_1 < k < k_0$, as

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u^+_\alpha(\mathbf{k},t) = M^+_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left[u^-_\beta(\mathbf{j},t)u^-_\gamma(\mathbf{k}-\mathbf{j},t)\right.$$

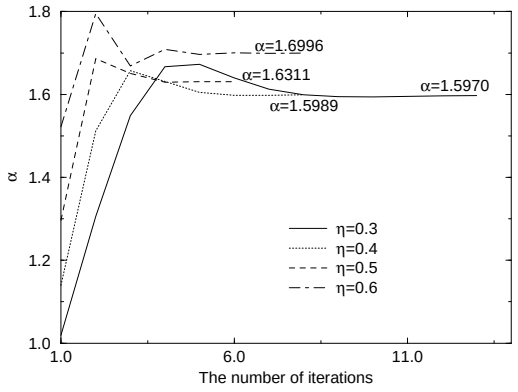

Figure 1: Convergence of the Kolmogorov spectral constant α to the fixed points for several values of the bandwidth parameter η.

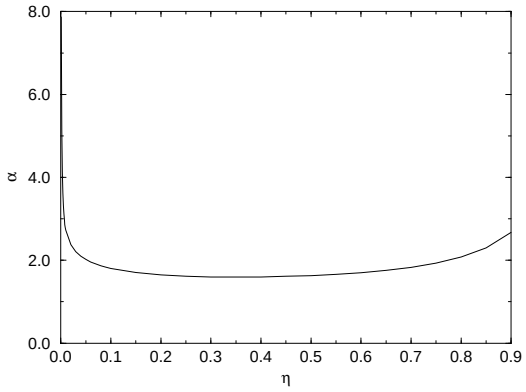

Figure 2: Dependence on the bandwidth parameter η of the calculated values of the Kolmogorov spectral constant α based on equation (6).

$$+ \quad 2u_\beta^-(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t) + u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\Big], \quad (4)$$

where the superscripts + and − on $M_{\alpha\beta\gamma}(\mathbf{k})$ have the same significance as for $u_\alpha(\mathbf{k},t)$, and the symmetrized inertial transfer operator $M_{\alpha\beta\gamma}(\mathbf{k}) = (2i)^{-1}[k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})]$.

In principle, the RG approach involves two stages: (i) Eliminate the high-k modes, $\mathbf{u}^+$, which appear in equation (3) for $0 < k < k_1$, by solving for the mean effect of the high-k field. This results in an increment to the viscosity, i.e. $\nu_0 \to \nu_1 = \nu_0 + \delta\nu_0$. (ii) Rescale the basic variables, so that the Navier-Stokes equation for $0 < k < k_1$ looks like the original Navier-Stokes equation for $0 < k < k_0$.

Although this procedure is appealingly simple and has a clear physical interpretation, it has not proved easy to put into practice in the turbulence problem. A typical approach is to eliminate all the high-k effects in equation (3), by substituting the solution of equation (4), directly into the $\mathbf{u}^+$ modes in the $\mathbf{u}^-$ equation. However, problems are then encountered because of the mode coupling between $\mathbf{u}^-$ and $\mathbf{u}^+$. Even if one succeeds in carrying out the first part, the further problem of averaging out the high-k modes arises immediately, because $\mathbf{u}^-$ and $\mathbf{u}^+$ are not statistically independent. This problem was avoided by Foster, Nelson and Stephen (1977; hereafter referred to as FNS) in their pioneering study of stirred fluid motion, as they restricted their attention to stirring forces which were multivariate normal and excluded the effects of the turbulence cascade. However, it has been shown that the use of a 'filtered' average by FNS to eliminate the $\mathbf{u}^-$ equation is really an uncontrolled approximation (Eyink, 1994).

Iterative-Averaging RG with Results

Here, we follow the method of iterative averaging, which is based upon the derivation of a recurrence relation and, in principle, eliminating finite blocks of modes (i.e. high-k modes) while maintaining the form invariance of the dynamical equation. Apart from the work of FNS, elimination procedures can be performed by 'conditional' averaging, first introduced by McComb (1982). Further details about the conditional average have been given elsewhere (McComb, Robert and Watt, 1992). The basic ansatz of a conditional average is that a small uncertainty (Φ^-, say) at the cutoff wavenumber will generate chaotic behaviour for the high-k modes. Although the introduction of Φ^- has been accepted, mainly due to the chaotic nature of the Navier-Stokes equations, it might be interesting to see how 'rapidly' chaotic behaviour develops from the given small Φ^- by numerical simulation. This aspect is one of our current tasks and the results will be reported in due course.

The current result of the iterative-averaging calculation for the Navier-Stokes equations after first eliminating the high-k effects is

$$\left[\frac{\partial}{\partial t} + \nu_1 k^2\right] u_\alpha^-(\mathbf{k},t) = M_{\alpha\beta\gamma}^-(\mathbf{k}) \int d^3j \; u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t), \quad (5)$$

where $\nu_1 = \nu_0 + \delta\nu_0(k)$ and

$$\delta\nu_0(k) = -\frac{1}{k^2}\int d^3j \; Q_v^+(\mathbf{j}) \times \frac{\frac{4}{d-1}\mathbf{Tr}\left[M_{\alpha\beta\gamma}^-(\mathbf{k})M_{\gamma\rho\sigma}^+(\mathbf{k}-\mathbf{j})D_{\beta\sigma}(\mathbf{j})\right]}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2}. \quad (6)$$

Here, we consider space dimension $d = 3$. This result can be extended to further shells, and we have shown elsewhere (McComb and Watt, 1992) that a fixed point is reached under numerical iteration of the recursion relations (see also Figure 1). In Figure 2, we show a calculation of the Kolmogorov constant $\alpha = 1.60 \pm 0.01$ independent of the bandwidth of modes being eliminated for bandwidths in the range $0.25 \le \eta \le 0.45$, in agreement with experiment.

NUMERICAL SIMULATIONS

Two programmes of numerical simulation are being carried out — one at the University of Edinburgh in the United Kingdom, the other at the Swiss Federal Institute of Technology, Lausanne. A large number of runs have already been carried out at Lausanne, and this paper presents some of the results obtained so far.

The simulations themselves are very similar, while the computer systems on which they are run differ greatly. At Edinburgh, work is carried out on a parallel machine, the Cray T3D, while in Lausanne a parallel-vector machine, the NEC SX-4, is used.

The simulations discussed in this paper were carried out at a resolution of 256^3, requiring approximately 14 seconds of SX-4 time per time-step on a single processor.

The general method of such simulations has been well established. We follow the work of Orszag for the construction of initial velocity fields (1969) and in the use of a pseudospectral method (1971). The time integration scheme is a second-order Runge-Kutta method and partial dealiasing is achieved by way of a random-shifting method (see, for example, Rogallo, 1981).

Δt	T	ν_o	k_f	k_0
10^{-3}	113.5	10^{-3}	1.5	120

ε	R_λ	L	λ	τ_E	s_3	s_4
.15	190.606	1.431	.246	1.853	$-.51$	6.053

Table 1: Characteristics of the simulation

Initial Conditions

The simulations are started with an initial energy spectrum of the form

$$E(k,0) = 16(2/\pi)^{\frac{1}{2}} u_0^2 k_{\rm p}^{-5} k^4 \exp[-2(k/k_{\rm p})^2] \qquad (7)$$

where $k_{\rm p}$ is the location of the spectrum's maximum and u_0 is the required initial r.m.s. velocity.

Forcing

Stationary turbulence is obtained by use of a deterministic forcing term

$$f_\alpha(\mathbf{k},t) = \begin{cases} \varepsilon u_\alpha(\mathbf{k},t)/(2E_f(t)) & \text{if } 0 < k < k_f, \\ 0 & \text{otherwise,} \end{cases} \qquad (8)$$

where ε is the mean dissipation rate, and

$$E_f(t) = \int_0^{k_f} E(k,t)\mathrm{d}k. \qquad (9)$$

There is no preferred direction in this forcing and the turbulence rapidly reaches a statistically isotropic and steady state.

Statistics

While our simulations are entirely conventional, we do not rely solely on the usual practice (as justified by isotropy) of averaging over shells in wavenumber space in order to obtain statistical quantities, but also generate many realizations in order to increase our sample size.

The main characteristics of the simulation are reported in Table 1, where Δt is the time step, T is the integration time, ν_o is the molecular viscosity, k_f is defined in (8), k_0 is the ultraviolet cut-off, ε is the mean dissipation rate, R_λ is the Reynolds number based on the Taylor microscale, L is the integral scale, λ is the Taylor microscale, τ_E is the turnover time and s_3 and s_4 are respectively the skewness and flatness of the velocity derivative.

The equations have been integrated for more than 60 turnover times and about 200 box-realizations of each component of the velocity field have been stored in a database. Since these box-realizations are separated by $\approx \tau_E/4$ they can be considered statistically independent for the middle-range-scales and the small-scales.

RESULTS

We wish to assess the freedom to carry out conditional averages of the type required by RG. In principle we may do this by extracting, from an ensemble of realizations of the velocity field

$$\mathcal{X} = \{X_\alpha^{(n)}(\boldsymbol{k},t)\,|\,\alpha = 1,2,3; t \in [0,T]; \\ 0 \le |\boldsymbol{k}| \le k_0; n = 1,...,N\}, \qquad (10)$$

two disjoint subensembles $\mathcal{Y}$ and $\mathcal{Z}$ chosen such that, for a prescribed $\zeta > 0$,

$$\frac{|\boldsymbol{Y}^{(m)}(\boldsymbol{k},t) - \boldsymbol{Z}^{(m)}(\boldsymbol{k},t)|^2}{2\langle|\boldsymbol{Y}^{(m)}(\boldsymbol{k},t)|^2\rangle} \le \zeta$$

$$\text{for all} \quad 0 \le |\boldsymbol{k}| \le k_c\,;\, m = 1,...,M\,;\, t \in [0,T], \qquad (11)$$

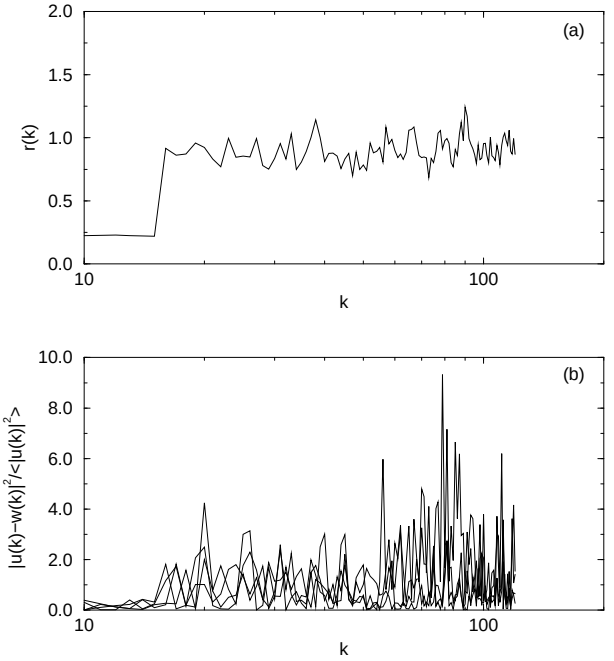

Figure 3: (a) Relative energy error for $k_b = 10$, $k_c = 15$, $\zeta = 0.5$ and $\alpha = 1$. (b) A selected set of realizations showing strong fluctuations for $k \ge 15$.

for each realization $\boldsymbol{Y}^{(m)} \in \mathcal{Y}$ and $\boldsymbol{Z}^{(m)} \in \mathcal{Z}$. We may then define the relative energy of the error

$$r(|\boldsymbol{k}|) = \frac{\langle(\boldsymbol{u}(\boldsymbol{k},t) - \boldsymbol{w}(\boldsymbol{k},t))^2\rangle}{2\langle\boldsymbol{u}(\boldsymbol{k},t)^2\rangle}, \qquad (12)$$

where $\boldsymbol{u}(\boldsymbol{k},t) \in \mathcal{Y}$ and $\boldsymbol{w}(\boldsymbol{k},t) \in \mathcal{Z}$. (It is important to note that the averages in the definition (12) are, in this context, subensemble averages defined on $\mathcal{Y}$ and $\mathcal{Z}$ and not ensemble averages on $\mathcal{X}$.) In equation (12) and subsequently, we assume that the fields are statistically stationary and isotropic, therefore r depends only on $|\boldsymbol{k}|$. Since the two fields are very close when $0 \le |\boldsymbol{k}| \le k_c$, $r(|\boldsymbol{k}|)$ will be much smaller than 1 in this interval, indicating that the fields are almost completely correlated. If the error between the fields grows in such a way that they become decorrelated, we will have $r(|\boldsymbol{k}|) \to 1$ as $|\boldsymbol{k}| \ge k_c$ increases.

In practice, our 200 box-realizations are not sufficient for the above analysis and we shall describe how we have extracted, using a partial sampling technique, enough realizations to compute the relative energy of the error defined by (12).

In order to this, we have performed the following partial Fourier transform of one component of the velocity field

$$u_\alpha(x,y,k) = \frac{1}{2\pi}\int u_\alpha(x,y,z)e^{ikz}\,dk, \qquad (13)$$

then we have selected, for each box-realization, a set of realizations, say $u_\alpha(x_i,y_i,k)$, where the spacing $\delta x = |x_{i+1} - x_i| = |y_{i+1} - y_i|$ is chosen such that the realizations are (approximately) independent for the range of k we consider (if we consider only the scales such that $k \ge k_b$, then $\delta x = 2\pi/k_b$). The union of all these realizations obtained for each of the box-realizations will constitute our ensemble $\mathcal{X}$. The subensemble $\mathcal{Y}$ is formed by choosing an arbitrary subensemble of $\mathcal{X}$. To select the subensemble $\mathcal{Z}$, we impose the condition

$$\frac{|Y^{(m)}(k) - Z^{(m)}(k)|^2}{2\langle|Y^{(m)}(k,t)|^2\rangle} \le \zeta$$

$$\text{for all} \quad k_b \le k \le k_c\,;\, m = 1,...,M. \qquad (14)$$

Note that the time dependence does not appear in the equations since all the box-realizations used to form the ensemble $\mathcal{X}$ are taken in the statistically steady regime. Figure 3(a) shows the relative energy error

$$r(k) = \frac{\langle (u(k) - w(k))^2 \rangle}{2\langle u(k)^2 \rangle}, \tag{15}$$

where $u \in \mathcal{Y}$ and $w \in \mathcal{Z}$ for $k_b = 10$, $k_c = 15$, $\zeta = 0.5$ and $\alpha = 1$. The number of realizations M is 2533. Though the number of realizations is not large enough to have a smooth converged solution, one can see that the relaxation to a chaotic regime is indeed very fast. Figure 3(b) shows a selected set of realizations for which one can observe that the constraint imposed for $10 \leq k \leq 15$ does not prevent strong fluctuations for $k \geq 15$. The convergence of $r(k)$ is difficult to improve, due to the restriction on the number of realizations available for a given constraint.

Another natural way in which the small-scale properties of a conditional subensemble may be investigated is by studying the probability density functions (pdfs) of velocity increments. In physical-space, we can use homogeneity in the three dimensions and have sufficiently large subensembles to compute high-order statistics and pdfs. The velocity increments are defined by the following relation

$$\delta \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{u}(\boldsymbol{x}), \tag{16}$$

where $\boldsymbol{h}$ is a displacement vector and $\boldsymbol{x}$ the position. Since the fields are statistically isotropic, we can restrict ourselves to the study of the longitudinal velocity increment $\delta v_L(h)$ which is the projection of $\delta \boldsymbol{u}(\boldsymbol{h})$ on the direction of the vector $\boldsymbol{h}$ and the lateral velocity increment $\delta v_T(h)$ which is the projection of $\delta \boldsymbol{u}(\boldsymbol{h})$ on a direction perpendicular to $\boldsymbol{h}$. For the purpose of this paper, we have only studied the longitudinal velocity increment $\delta v_L(h)$. We have selected two scales, $h_1 = \lambda/1.26$ and $h_2 = \lambda/5.01$ (λ is the Taylor micro-scale, therefore h_1 is a typical scale in the inertial subrange and h_2 is in the dissipation subrange). The selection of the subensembles is performed using conditions of the type $a < \delta v_L(h_1) < b$. The pdfs of $\delta v_L(h_2)$ for the unconditional ensemble and for the subensembles are then compared. Figure 4 gives the normalized pdf (σ is the standard deviation of $\delta v_L(h)$) of the unconditional ensemble for $h = h_1$ and $h = h_2$. We observe the classical result that the tails of the pdfs are growing as the scale is decreased which is the signature of growing intermittency. The pdf also shows a negative skewness which is a direct consequence of the nonlinear dynamics of the Navier-Stokes equations. Figure 5, shows the pdfs of the unconditional ensemble and of a subensemble defined by the constraint $-1 < \delta v_L(h_1) < 0$. The pdfs are almost superimposed, showing that the flow at scale h_2 is unaffected by the condition imposed at scale h_1. Figure 6 is a case for which the subensemble is much smaller due to a more restrictive condition, $1 < \delta v_L(h_1) < 4$. However, the general behavior of the pdf supports the view that the chaotic dynamics of the Navier-Stokes equations tends to restore the original distribution. Note that the skewness is incorrectly predicted and seems to be correlated with the sign of $\delta v_L(h_1)$. Figure 7 presents a case with a very strong condition, $-7 < \delta v_L(h_1) < -2$. Though the number of realizations is small, we observe that the top of the pdf is quite accurately reproduced.

CONCLUSION

These results, although preliminary in nature, offer crucial support to the hypothesis that a conditional average may be used to reduce the number of degrees of freedom required for the numerical simulation of turbulence. Work is continuing to make a more stringent assessment of the validity of such averages for turbulence and this includes

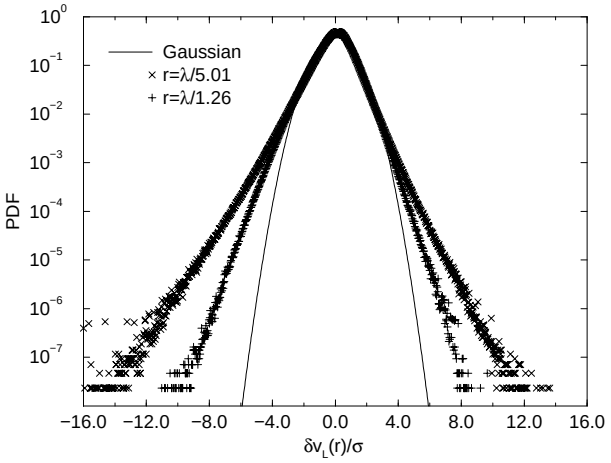

Figure 4: Normalized pdf of the unconditional ensemble.

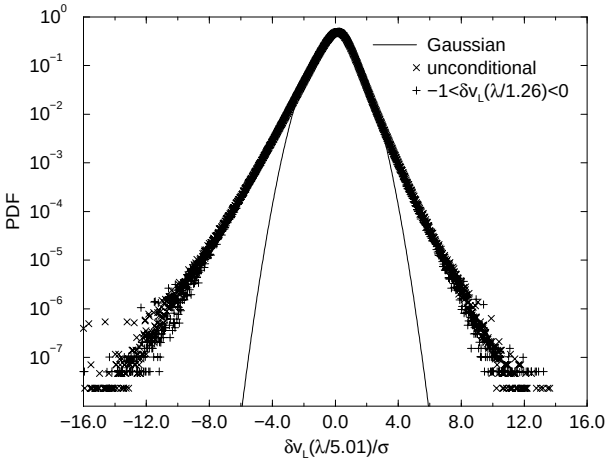

Figure 5: The pdfs of the unconditional ensemble and a subensemble defined by the constraint $-1 < \delta v_L(h_1) < 0$.

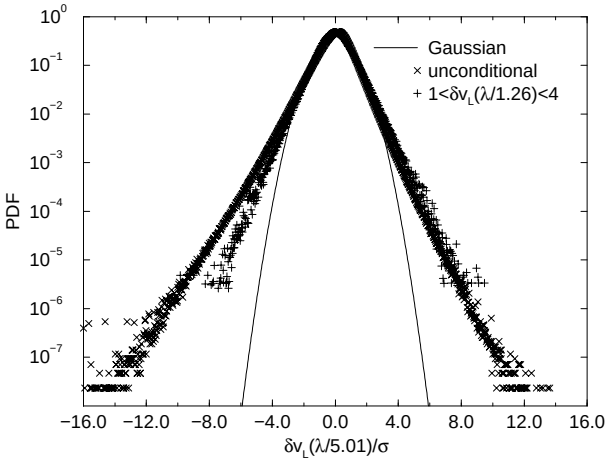

Figure 6: The pdfs of the unconditional ensemble and a subensemble defined by the constraint $1 < \delta v_L(h_1) < 4$.

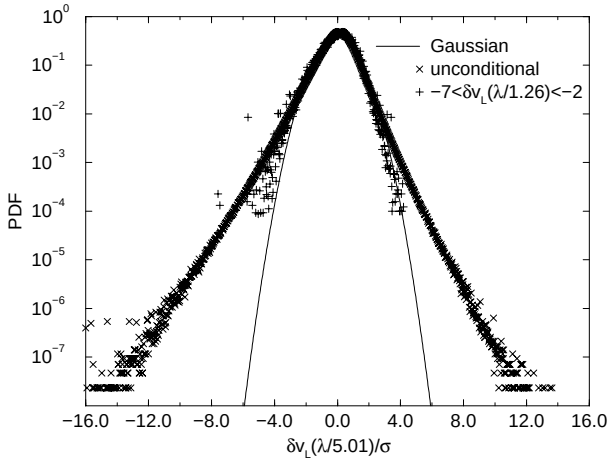

Figure 7: The pdfs of the unconditional ensemble and a subensemble defined by the constraint $-7 < \delta v_L(h_1) < -2$.

carrying out simulations at higher numerical resolution. At present we are working on a 512^3 simulation and hope to present results from this at the conference.

Acknowledgements: The simulation presented in this paper has been performed on the computers of the Swiss Center for Scientific Computing, Manno. The research of L. Machiels is supported by the Swiss National Foundation for Scientific Research.

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